SOLVING THE CONSUMER PROBLEM: EXAMPLES

ADLAI NEWSON

example 1: Cobb-Douglas. Solve for the optimal consumer choices given preferences

$$u(x,y) = \alpha x^{\beta} y^{1-\beta}$$

and budget constraint $m = p_y y + p_x x$.

Solution:

This is a slightly more general budget constraint than we used in discussion. Again, using the argument that the solution can't be at the boundary (suppose x or y is zero, then my utility is zero, but I can get positive utility by increasing whichever good is zero by a tiny amount), we will use the equation that works whenever we are dealing with an interior solution: $MRS = \frac{p_x}{p_y}$. Since $MRS \equiv \frac{u_x}{u_y}$, let's first find these two derivatives:

$$\begin{aligned} u_x &= \alpha\beta x^{\beta-1} y^{1-\beta} \\ &= \alpha\beta \left(\frac{y}{x}\right)^{1-\beta} \\ u_y &= \alpha(1-\beta) x^{\beta} y^{-\beta} \\ &= \alpha\left(1-\beta\right)\left(\frac{x}{y}\right)^{\beta} \end{aligned}$$

Dividing these two to get the MRS, we have that

$$\begin{aligned} MRS &= \frac{u_x}{u_y} \\ &= \frac{\beta}{1-\beta}\left(\frac{y}{x}\right) \end{aligned}$$

Notice that the α cancelled out, as we argued it must in tutorial. Now we set MRS equal to the price ratio $\frac{p_x}{p_y}$ to obtain

$$\begin{aligned} \frac{\beta}{1-\beta}\left(\frac{y}{x}\right) &= \frac{p_x}{p_y} \\ y &= \left(\frac{1-\beta}{\beta}\right)\frac{p_x}{p_y} x \end{aligned}$$

In order to solve for x and y we use the budget constraint,

$$\begin{aligned} m &= p_x x + p_y y \\ &= p_x x + p_y \left[\left(\frac{1-\beta}{\beta} \right) \frac{p_x}{p_y} x \right] \\ &= p_x x \left(1 + \frac{1-\beta}{\beta} \right) \\ &= \frac{p_x x}{\beta} \end{aligned}$$

rearranging, we get:

$$x = \frac{m\beta}{p_x}$$

and using this in the equation above for y, we get

$$y = \frac{(1-\beta)\,m}{p_y}$$

These are the general solutions to Cobb-Douglas preferences. Notice that the optimal choices for x and y depend only on their own prices, their own exponent from the utility function, and income. Notice also that if we divide x by y we get:

$$\frac{x}{y} = \frac{\beta}{1-\beta} \frac{p_y}{p_x}$$

and income is gone! We have just shown that while the amount of x, y will change as income changes, the ratio x/y is the same for all income levels.

example 2: Quasi-linear. Solve for the optimal consumer choices given preferences

$$u(x,y) = x + \beta \log y$$

and budget constraint $m = p_x x + p_y y$.

Solution:

The first question we should ask is whether the solution will neccessarily be interior. Can we use the same logic as we used in the first example? Sort of. Notice that if $y = 0$ then $\log y = -\infty$, so no matter how much x I get, I still get $-\infty$ utility (because $-\infty$ + any number is still $-\infty$). That means I don't need to worry about the boundary case where I don't consume any y. But what about $x = 0$? Because I will be getting some utility from y, it's not clear that we can rule out $x = 0$.

Let's see how far we get using $MRS = \frac{p_x}{p_y}$. First, the derivatives:

$$\begin{aligned} u_x &= 1 \\ u_y &= \frac{\beta}{y} \end{aligned}$$

So that

$$MRS = \frac{p_x}{p_y}$$
$$\frac{y}{\beta} = \frac{p_x}{p_y}$$
$$y = \beta \frac{p_x}{p_y}$$

but this isn't a function of x! Let's keep going, and use the budget constraint to get x:

$$m = p_x x + p_y y$$
$$= p_x x + p_y \left(\beta \frac{p_x}{p_y}\right)$$
$$x = \frac{m}{p_x} - \beta$$

It looks like we've solved it, but does this answer for x really make sense? What if β is huge.. then our answer says that x will be negative, which is impossible. From our answer for x, in order to have $x > 0$, we need that

$$\beta < \frac{m}{p_x}$$

If this is true, then we have found the answer. If it's not true, then we have $x = 0$, and all income is spent on y, so $y = \frac{m}{p_y}$. This was an example of quasi-linear preferences (quasi-linear because preferences are linear in one good and nonlinear in the other). It's a general feature of quasilinear preferences that we will find the answer for the nonlinear good, and the linear good will be found as a 'residual' - ie, we will spent all the leftover income on it.

example 3: Perfect Substitutes. Solve for the optimal consumer choices given preferences

$$u(x,y) = x + \beta y$$

and budget constraint $m = p_x x + p_y y$.

Solution

Now it's totally unclear if the solution will be interior. Again let's see what happens if we use the $MRS = \frac{p_x}{p_y}$ equation:

$$\frac{u_x}{u_y} = \frac{p_x}{p_y}$$
$$\frac{1}{\beta} = \frac{p_x}{p_y}$$

But we can't use this to solve for either x or y. In fact, it's just an equation in exogenous parameters. Let's suppose it just happens to be true (for example, $\beta = 1$, $p_x = 1$, $p_y = 1$). Then because the equation that gives us our optimal choice of x, y is satisfied for *any* x, y that also satisfies the budget constraint. Take a look at the utility function.. when $\beta = 1$ we see that x and y give identical utility, and $p_x = p_y = 1$ means x and y cost the same. That means I'm indifferent between any combination of x and y: I get the same utility. Another way of saying this is that the highest indifference curve (which has slope $-\frac{1}{\beta}$) perfectly overlaps the budget line (which has slope $-\frac{p_x}{p_y}$).

Now consider the case $\frac{1}{\beta} < \frac{p_x}{p_y}$. First think about it intuitively; this means that β is "too large" relative to prices. Looking at the utility function, we would guess that if β is large, the agent will spend all his money on y. The technical way of saying the same thing is that the slope of the indifference curve $-\frac{1}{\beta}$ is closer to zero than $-\frac{p_x}{p_y}$, so the budget line is too steep. If you draw this in x, y space you'll see that the highest indifference curve will intersect at the y intercept. So in this case our solution is $x = 0$, $y = \frac{m}{p_y}$.

For the case $\frac{1}{\beta} > \frac{p_x}{p_y}$ the logic is identical, but now it's the indifference curves that are too steep, which means the highest IC intersects the budget line at the x intercept; $x = \frac{m}{p_x}$, $y = 0$.

Another way of getting this answer would have been to substitute for x using the budget line: $x = \frac{m}{p_x} - \frac{p_y}{p_x}y$, and consider the three cases:

(1) $u(x,y) = \frac{m}{p_x} - \frac{p_y}{p_x}y + \beta y$ when $x \geq 0$, $y \geq 0$
(2) $u(0, \frac{m}{p_y}) = \beta \frac{m}{p_y}$
(3) $u(\frac{m}{p_x}, 0) = \frac{m}{p_x}$

Then check all the inequalities to see which utility is the biggest. For example, for $x > 0$, $y = 0$ to be better than choosing $x > 0$, $y > 0$ it must be true that

$$
\begin{aligned}
u(\frac{m}{p_x}, 0) &> u(x,y) \\
\frac{m}{p_x} &> \frac{m}{p_x} - \frac{p_y}{p_x}y + \beta y \\
\frac{p_y}{p_x}y &> \beta y \\
\frac{p_y}{p_x} &> \beta \\
\frac{p_x}{p_y} &< \frac{1}{\beta}
\end{aligned}
$$

which is the same condition we found above.

Remark. Is my example of perfect substitutes limited because there's no coefficient on x? ie, if you see $\tilde{u}(x,y) = ax + by$ should you throw the example above out the window? Remember that I can do positive monotonic transformations of the utility function, so if I define $u = \frac{1}{a}\tilde{u}$, I get $u(x,y) = x + \frac{b}{a}y$. Now define $\beta = \frac{b}{a}$ and the example above holds.

example 4: Perfect compliments. Solve for the optimal consumer choices given preferences

$$u(x,y) = \min\{x, \beta y\}$$

and budget constraint $m = p_x x + p_y y$.

Solution

This is a reasonably simple case of perfect compliments preferences. The key to solving these questions is realizing that the optimal choice *must be at the kink in the indifference* *curve*. In particular, where the kink in the curve touches the budget line. Draw a few examples and see if you can argue why this is true. Even though the solution is interior for sure (why? see if you can argue just from the utility function), we can't use the $MRS = \frac{p_x}{p_y}$ equation because the indifference curve is non-differentiable at the kink. If you draw a few indifference curves you might notice that the kink will always be where $x = \beta y$. This immediately suggests how to solve the question:

(1) Set the arguments of the min function equal to each other (in this case, $x = \beta y$); this is the line where all the kink-points will be.
(2) Find the point on the kink-point line $x = \beta y$ that intersects the budget line. This will be the optimal choice.

How do we find the intersection of two lines? Just by solving for x, y that satisfy both equalities.

$$\begin{aligned} m &= p_x x + p_y y \\ &= p_x(\beta y) + p_y y \\ &= y(\beta p_x + p_y) \\ y &= \frac{m}{\beta p_x + p_y} \\ x &= \frac{\beta m}{\beta p_x + p_y} \end{aligned}$$

Done!