1. Consider a game between country U (representing the interests of most nations) and country T. There is a concern that country T is developing nuclear weapons. Country U has demanded that country T cease any weapons program. Country T may be of two types: passive, T_P or aggressive, T_A. The probability distribution over types is common knowledge with $\mathbb{P}(T_P) = \frac{3}{4}$. Country T must choose whether to Heed (H) the demands of country U or Ignore (I) the demands of country U. Country U will observe the choice of country T. If T heeds the demands then the game ends with payoffs $(U, T_A) = (9, 1)$ and $(U, T_P) = (9, 7)$. If country T chooses to ignore the demands of country U, this will be observed by country U and U must then choose whether to Maintain (M) their resolve (continue to demand that the weapons program be stopped) or Back Down (B). If Country U chooses to back down then the game will end with payoffs $(U, T_A) = (5, 8)$ and $(U, T_P) = (5, 10)$. If country U chooses to maintain resolve then country T will observe this and must choose whether to wage war (W) or no war (N). Payoffs from war will be $(U, T_A) = (3, 5)$ and $(U, T_P) = (3, 1)$, payoffs from no war will be $(U, T_A) = (6, 3)$ and $(U, T_P) = (6, 5)$.

   (a) Draw the game tree under the assumption of full-information (country U is able to observe the type of country T).

   (b) Find the subgame perfect Nash equilibrium to this game. What are the beliefs of country U in this equilibrium?

   (c) Draw the game tree under the assumption that country U is unable to observe the type of country T.

   (d) Are there any pure strategy perfect Bayes-Nash equilibria to this game?

   (e) Would the type T_A country ever mix over Heed and Ignore? Would the type T_P country ever mix over Heed and Ignore?

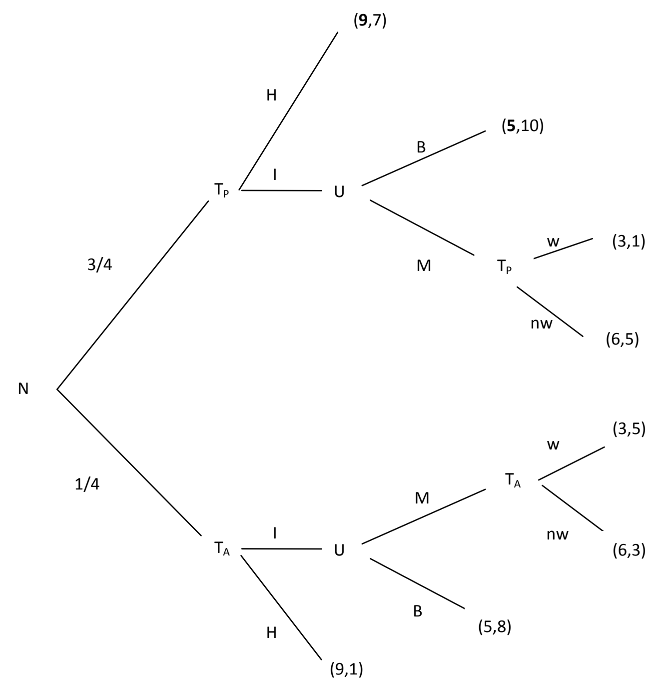
   (f) If country T_P randomizes over H and I with probability α on playing H, derive expressions for the beliefs of country U regarding what type of country they face if T chooses to ignore.

   (g) Find a mixed strategy equilibrium where T_P mixes over H and I and country U mixes over M and B. Describe strategies and beliefs (of country U).

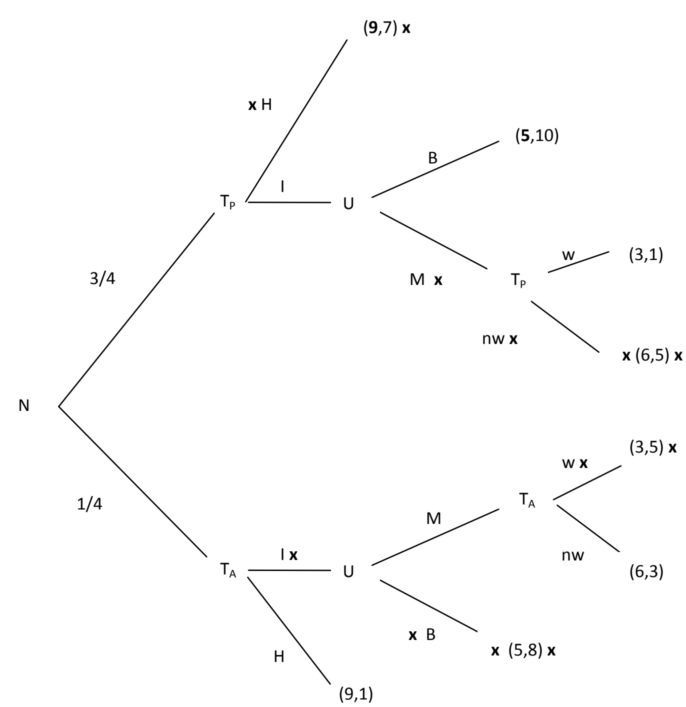
Solutions

1. incomplete information I

(a)



(b) Find the subgame perfect Nash equilibrium to this game. What are the beliefs of country U in this equilibrium?



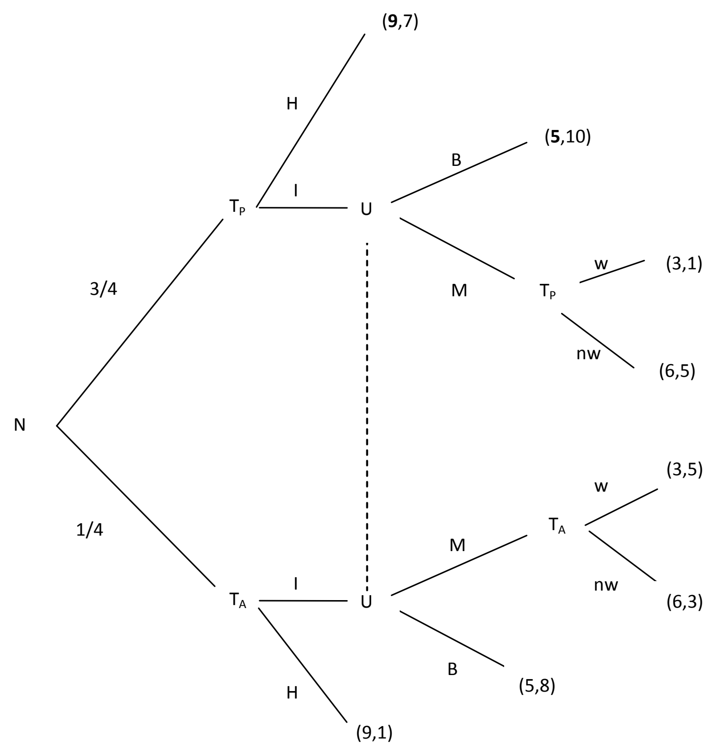
U plays M if facing type T_P and B if facing type T_A

T_P plays (H, nw)

T_A plays (I, w)

For player U, they are not uninformed, they will form beliefs based on observed type.

(c) Draw the game tree under the assumption that country U is unable to observe the type of country T.



(d) Are there any pure strategy perfect Bayes-Nash equilibria to this game?

To check for a pure strategy equilibrium we can first of all consider type T_A. Type T_A will never want to play H since they can always do better by playing I no matter what player U will do. Type T_A will also always play w. Any pure strategy equilibrium must have type T_A playing the strategy (I, w).

For type T_P, it is clear that a pure strategy must have this player playing nw. The possible pure strategies of type T_P are (H, nw) and (I, nw).

There are 4 pure strategy profiles (player U, player T_P, player T_A) that we should consider:

$$\{M, (H, nw), (I, w)\}$$
$$\{B, (H, nw), (I, w)\}$$
$$\{M, (I, nw), (I, w)\}$$
$$\{B, (I, nw), (I, w)\}$$

We can check each of these to see if any player (player U or type T_P have an incentive to deviate).

$\{M,(H,nw),(I,w)\}$ This can't be an equilibrium because if type T_P is playing H then player U will form the belief $\mathbb{P}(T_A|I)=1$ and want to play B when a player has played I.

$\{B,(H,nw),(I,w)\}$ This can't be an equilibrium because if player U is playing B then T_P will want to play I.

$\{M,(I,nw),(I,w)\}$ This can't be an equilibrium because if U is playing M then T_P will want to play H.

$\{B,(I,nw),(I,w)\}$ This can't be an equilibrium because if both T_P and T_A are playing I then U will want to play M.

There is no pure strategy equilibrium.

(e) Would the type T_A country ever mix over Heed and Ignore? Would the type T_P country ever mix over Heed and Ignore?

Type T_A will never mix. Type T_P will mix over H and I if $\mathbb{E}\{H\}=\mathbb{E}\{I\}$.

(f) If country T_P randomizes over H and I with probability α on playing H, derive expressions for the beliefs of country U regarding what type of country they face if T chooses to ignore.

Let T_P play H with probability α and I with probability $(1-\alpha)$. The beliefs of country U on observation of I will be:

$$\begin{aligned}\mathbb{P}_U(T_P|I) &= \frac{\mathbb{P}(I|T_P)\mathbb{P}(T_P)}{\mathbb{P}(I|T_P)\mathbb{P}(T_P)+\mathbb{P}(I|T_A)\mathbb{P}(T_A)} \\ &= \frac{(1-\alpha)\frac{3}{4}}{(1-\alpha)\frac{3}{4}+(1)\,\frac{1}{4}} \\ &= \frac{3-3\alpha}{4-3\alpha}\end{aligned}$$

and

$$\mathbb{P}_U(T_A|I)=1-\mathbb{P}_U(T_P|I)=\frac{1}{4-3\alpha}$$

(g) Find a mixed strategy equilibrium where T_P mixes over H and I (probabilities α and $1-\alpha$) and country U mixes over M and B (probabilities β and $1-\beta$). Describe strategies and beliefs (of country U).

for U :

$$\begin{aligned}\mathbb{E}_U\{B\} &= \mathbb{E}_U\{M\} \\ 5 &= 6\mathbb{P}_U(T_P|I)+3\mathbb{P}_U(T_A|I) \\ 5 &= 6\mathbb{P}_U(T_P|I)+3(1-\mathbb{P}_U(T_P|I)) \\ 2 &= 3\mathbb{P}_U(T_P|I) \\ \mathbb{P}_U(T_P|I) &= \frac{2}{3} \\ \frac{1}{4-3\alpha} &= \frac{1}{3} \\ 4-3\alpha &= 3 \\ \alpha &= \frac{1}{3}\end{aligned}$$

for T_P :

$$\begin{aligned}\mathbb{E}\{H\} &= \mathbb{E}\{I\}\\ 7 &= 10(1-\beta)+5\beta\\ \beta &= \frac{3}{5}\end{aligned}$$

beliefs

$$\begin{aligned}\mathbb{P}_U(T_P|I) &= \frac{2}{3}\\ \mathbb{P}_U(T_A|I) &= \frac{1}{3}\end{aligned}$$

player U plays M with probability $\beta = \frac{3}{5}$, B with probability $\frac{2}{5}$
type T_P plays H with probability $\alpha = \frac{1}{3}$ and I with probability $\frac{2}{3}$ and nw
type T_A plays I and w